TWO NOTES ON THE PROBABILITY OF RADIATIVE TRANSITIONS

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Abstract

In 1 we compute the rate at which electrons and protons should, on Dirac's theory of electrons and protons, annihilate each other; this gives a mean life time for matter of the order of 10^{-10} sec.

In 2 we compute by Dirac's radiation theory the relative probability of radiative and radiationless transitions; we obtain an expression substantially equivalent to that derived by Heisenberg and Pauli.

I

An electron satisfying Dirac's linear wave equation will very rapidly lose energy to the electromagnetic field. If the electron is free, it must lose this energy by radiating at least two quanta, in order that energy and momentum may be conserved in the process. If the electron is bound, e.g. in an atom, transitions in which only one quantum is radiated can occur, since there are then other particles present which can take up the necessary momentum. But these transitions are rare compared with the two-quantum transitions, which, as is well known, may be expected according to the theory to occur at an infinite rate. Now Dirac has suggested[1] that the reason why these transitions do not in fact occur is that the states of negative energy are filled; and this suggestion leads to a satisfactory understanding of the validity of the scattering laws derived from his wave equation. But according to Dirac not all of the states of negative energy are full; there are a few gaps in the distribution for negative electrons nearly at rest; and thus transitions to states of negative energy should not be quite excluded. Dirac further suggests that the empty states of negative energy are protons; and thus the filling of these states should correspond to the annihilation of an electron and a proton. This should occur very rarely; and if Dirac's suggestion were correct, we should expect to find a very small value for the corresponding transition probability. In this note we shall compute this transition probability on the basis of the present theory.

This computation cannot be made theoretically unique and certain until the grave difficulties introduced by the inequality of electron and proton masses are resolved; and this resolution seems to demand an essential advance[2] in the theory. The chief ambiguity for the present work arises from the fact that the energy radiated by the conversion of a stationary positive elec-

[1] P. A. M. Dirac, Roy. Soc. Proc. **A126**, 360 (1930).

[2] J. R. Oppenheimer, Phys. Rev. **35**, 461 (1930).

tron into a stationary negative electron is $2\ mc^2$; whereas the energy liberated by the annihilation of a stationary electron and a stationary proton should be $(m+M_P)c^2$. We shall make the computation without explicit recognition of the difference in mass of electron and proton; this gives a transition probability which is absurdly large, and which is not appreciably reduced by the substitution of $m+M_P$ for $2m$ in the final formula.

Let us consider for definiteness an enclosure of volume V in which there is one free electron, and one gap in the negative energy distribution; and let both the electron and the gap be at rest. If electron and proton have in fact a relative velocity v, then our result will be in error by terms of the relative order $(v/c)^2$. For the wave equation of the electron we write

$$\{W/c + \alpha_0 mc + (h/2\pi i)[\alpha_1(\partial/\partial x) + \alpha_2(\partial/\partial y) + \alpha_3(\partial/\partial z]\}\psi = 0 \qquad (1)$$

We take the matrices $||\alpha_\mu(\rho\sigma)||$ in the form

$$||\alpha_1(\rho\sigma)|| = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}; \quad ||\alpha_2(\rho\sigma)|| = \begin{Vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{Vmatrix}$$
$$||\alpha_3(\rho\sigma)|| = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}; \quad ||\alpha_0(\rho\sigma)|| = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \qquad (2)$$

The normalized solutions corresponding to momenta p, q, r, and energy W given by

$$(W/c)^2 = m^2c^2 + p^2 + q^2 + r^2 \qquad (3)$$

may then be written

$$\psi_3{}^\alpha = 0;$$
$$\psi_4{}^\alpha = \frac{(mc + W/c)e^{2\pi i/h(px+qy+rz)}}{[(mc + W/c)^2 + p^2 + q^2 + r^2]^{1/2}V^{1/2}}$$
$$\psi_1{}^\alpha = \frac{-(p + iq)e^{2\pi i/h(px+qy+rz)}}{[(mc + W/c)^2 + p^2 + q^2 + r^2]^{1/2}V^{1/2}};$$
$$\psi_2{}^\alpha = \frac{re^{2\pi i/h(px+qy+rz)}}{[(mc + W/c)^2 + p^2 + q^2 + r^2]^{1/2}V^{1/2}}$$

and

$$\psi_4{}^\beta = 0; \qquad \psi_3{}^\beta = \psi_4{}^\alpha$$
$$\psi_1{}^\beta = -\psi_2{}^\alpha; \quad \psi_2{}^\beta = \tilde{\psi}_1{}^\alpha. \qquad (4)$$

The two wave functions for α and β give electrons with spin oriented parallel and antiparallel respectively to the z axis.

For the initial state of the electron we take

$$W = E_0 = h\nu_0 = mc^2;\ \psi_1{}^0 = \psi_2{}^0 = 0;\ \psi_3{}^0 = e^{2\pi i\gamma}(2V)^{-1/2};\ \psi_4{}^0 = (2V)^{-1/2} \quad (5)$$

similarly for the final state we take

$$\begin{aligned} &W = E^e = -\ h\nu_0 = -\ mc^2; \quad \psi_3{}^e = \psi_4{}^e = 0; \\ &\psi_1{}^e = e^{2\pi i\delta}\,(2V)^{-1/2}; \qquad \psi_2{}^e = (2V)^{-1/2}. \end{aligned} \quad (6)$$

Here γ and δ are independent indeterminate phases; if in the final results we average over these, we shall have assumed random orientations for the spins of both initial and final states; and with this understanding both (5) and (6) may be regarded as spherically symmetric.

We shall need also the wave functions for states with

$$p = q = 0;\ r = \pm\ mc;\ W = \pm\ (2)^{1/2}mc^2.$$

These are

$$\psi_2{}^\alpha = \psi_4{}^\alpha = 0;\ \psi_1{}^\alpha = \frac{-\ re^{\pm 2\pi i mcz/h}}{mcV^{1/2}(4 \pm 2(2)^{1/2})^{1/2}};\ \psi_3{}^\alpha = \frac{(1 \pm (2)^{1/2})e^{\pm 2\pi i mcz/h}}{V^{1/2}(4 \pm 2(2)^{1/2})^{1/2}}$$

and

$$\psi_1{}^\beta = \psi_3{}^\beta = 0;\ \psi_2{}^\beta = -\ \psi_1{}^\alpha;\ \psi_4{}^\beta = \psi_3{}^\alpha. \quad (7)$$

Initially there is to be no radiation present; the electron is in the state (0). The probability amplitude at a later time t that a quantum of frequency ν, momentum $\boldsymbol{p}$, and electric vector polarized along the unit vector $\boldsymbol{\varepsilon}$, shall have been emitted, and that the electron shall have jumped to a state of momentum $\boldsymbol{P}$, energy $h\bar{\nu}$, and polarization of spin $\tau = \alpha, \beta$, is then

$$\phi(\boldsymbol{p},\boldsymbol{\varepsilon};\ \boldsymbol{P},\bar{\nu},\tau) = -\ v(0;\boldsymbol{p},\boldsymbol{\varepsilon},\bar{\nu},\tau)\frac{1 - e^{2\pi i(\nu+\bar{\nu}-\nu_0)t}}{\nu + \bar{\nu} - \nu_0}\cdot \quad (8)$$

Here $v(o;\ \boldsymbol{p},\ \boldsymbol{\varepsilon},\ \bar{\nu},\ \tau)$ is the matrix component for the transition in question of the interaction energy between the electron and the light quantum field. This vanishes except when $\boldsymbol{P} = -\boldsymbol{p}$, and for $\boldsymbol{P} = -\boldsymbol{p}$ has the value

$$ec(2\pi/\nu hV)^{1/2}\left(\sum_{l=1}^{3}\epsilon_l\alpha_l e^{-2\pi i/h(\mathbf{P}.\,\mathbf{r})}\right)0;\ \bar{\nu},P,\tau;\text{with } \bar{\nu}^2 = \nu_0{}^2 + \nu^2 \quad (9)$$

where

$$\left(\sum_{l=1}^{3}\epsilon_l\alpha_l e^{-2\pi i/h(\mathbf{P}.\,\mathbf{r})}\right)0;\ \bar{\nu},P,\tau$$

is the matrix component corresponding to the electronic transition

$$(0) \rightarrow (\bar{\nu},P,\tau)$$

of the component of the vector

$$\boldsymbol{\alpha}e^{-2\pi i/h(\mathbf{P}.\,\mathbf{r})}$$

parallel to $\boldsymbol{\varepsilon}$.

The probability amplitude that the electron should have jumped to state (e), and that a second quantum of frequency ν', momentum $\boldsymbol{p}'$, and polarization along $\boldsymbol{\varepsilon}'$ should have been emitted, vanishes except when $\boldsymbol{p}'=-\boldsymbol{p}=\boldsymbol{P}$, and for $\boldsymbol{p}'=\boldsymbol{P}$ has the value

$$\phi(\boldsymbol{p},\boldsymbol{\varepsilon},\boldsymbol{p}',\boldsymbol{\varepsilon}';\,e)=\frac{2\pi e^2c^2}{h\nu V}\sum\left\{\frac{1-e^{2\pi i(\nu-\bar{\nu}-\nu_0)t}}{\nu-\bar{\nu}-\nu_0}-\frac{1-e^{4\pi i(\nu-\nu_0)t}}{2(\nu-\nu_0)}\right\}$$

with

$$\sum=\sum_{\bar{\nu}}\sum_{t}\Bigg\{\left(\sum_{e=1}^{3}\epsilon_l\alpha_l e^{-2\pi i/h(\boldsymbol{p}.\boldsymbol{r})}\right)_{0;\bar{\nu},P\tau,}\left(\sum_{l=1}^{3}\epsilon_l'\alpha_l e^{2\pi i/h(\boldsymbol{p}.\boldsymbol{r})}\right)_{\nu,P,\tau;e}$$

$$+\left(\sum_{l=1}^{3}\epsilon_l'\alpha_l e^{2\pi i/h(\boldsymbol{p}.\boldsymbol{r})}\right)_{0;0,P,\tau}\left(\sum_{l=1}^{3}\epsilon_l\alpha_l e^{-2\pi i/h(\boldsymbol{p}.\boldsymbol{r})}\right)_{0,P\tau;e}\Bigg\}(\nu+\bar{\nu}-\nu^{-1})\tag{10}$$

This grows large only for $\nu\sim\nu_0$, $\bar{\nu}\sim\pm(2)^{1/2}\nu_0$; and here the first term in the bracket may be neglected.

To evaluate $\sum$ for $\nu=\nu_0$, $\bar{\nu}=\pm(2)^{1/2}\nu_0$, we may without loss of generality take $\boldsymbol{p}$ along z, since both initial and final wave functions are effectively spherically symmetric. We may further take, again without loss of generality, $\boldsymbol{\varepsilon}$ along x. There are then two cases to consider, with $\boldsymbol{\varepsilon}'$ along x and along y respectively. For these cases we have in turn

$$\sum=\sum_{\bar{\nu}=\pm(2)^{1/2}\nu_0}\sum_{\tau=\alpha,\beta}\bar{\nu}^{-1}\Bigg\{\Bigg[\int dV\sum_{p,\sigma}\bar{\psi}_\sigma^{\bar{\nu},P,\tau}e^{-2\pi i/c.\nu_0 z}\alpha(\sigma\rho)\psi_\rho^{0}\Bigg\}$$

$$\left\{\int dV\sum_{\rho\sigma}\psi_\sigma^{e}e^{2\pi i/c\nu_0 z}\begin{Bmatrix}\alpha_1(\sigma,\rho)\\ \alpha_2(\sigma,\rho)\end{Bmatrix}\psi_\rho^{\bar{\nu},P,t}\right\}\tag{11}$$

$$+\left\{\int dV\sum_{\rho\sigma}\bar{\psi}_\sigma^{\nu,P\tau}e^{2\pi i/c.\nu_0 z}\begin{Bmatrix}\alpha_1(\sigma,\rho)\\ \alpha_2(\sigma\,,\,\rho\end{Bmatrix}\psi_\rho^{0}\right\}\left\{\int dV\sum_{\rho\sigma}\bar{\psi}_\sigma^{e}e^{-2\pi i/c\cdot\nu_0 z}\alpha_1(\sigma,\rho)\psi_\rho^{\bar{\nu},P,\tau}\right\}\Bigg].$$

If we use (2) and (7) this gives us

$$\begin{aligned}\sum&=0\ \text{ for }\boldsymbol{\varepsilon}'\|x\\ \sum&=(-\,i/2\nu_0)(1+e^{2\pi i(\gamma-\delta)})\ \text{ for }\boldsymbol{\varepsilon}'\|y.\end{aligned}\tag{12}$$

From this we conclude that the probability of an emission is proportional to the square of the sine of the angle between the electric vectors of the two quanta.

Now there are $2\nu_0^2(V/c^3)$ components of the radiation field per unit solid angle per unit frequency about ν_0; the direction of propagation of one quantum can vary over a hemisphere; but when this is fixed polarization, frequency and direction of propagation of the other quantum are determinate. Thus we get the total chance of an emission at time t by integrating the absolute value of the square of ϕ (averaged over γ and δ) over a hemisphere of solid angle and all frequencies, and multiplying by $2\nu_0^2(V/c^2)$; thus

$$\sum_{p,\epsilon} \int d\gamma \int d\delta \,|\, \phi(\boldsymbol{p},\boldsymbol{\epsilon}, \; -\boldsymbol{p},\boldsymbol{\epsilon}'; e) \,|^{2}$$

$$= \frac{4\pi\nu_0{}^2V}{c^3} \frac{e^4c^4 \cdot 4\pi^2}{h^2\nu_0{}^2V^2} \cdot \frac{1}{2\nu_0{}^2} \cdot \int_0^\infty \frac{d\nu \,|\, 1 - e^{4\pi i(\nu-\nu_0)t} \,|^2}{4(\nu - \nu_0)^2}$$

$$= t \cdot \frac{16\pi^4 e^4 c}{h^2\nu_0{}^2 V} \cdot \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx \tag{13}$$

$$= t \cdot \frac{16\pi^5 e^4}{m^2c^3V} \cdot$$

The mean life time of an electron in a proton density of n_P protons per unit volume is thus

$$T = \frac{m^2c^3}{16\pi^5e^4n_P} \sim \frac{5\times 10^{10}}{n_p} \text{ sec.} \tag{14}$$

It should be observed that the retention of the terms for $\bar{\nu} = -2^{1/2}\nu_0$ in the expression (11) for $\sum$, may be justified by an argument similar to that used by Dirac to validate the scattering formulae. For although the electron cannot jump to this state of negative energy, because it is already filled, there is a double transition which gives just the same terms in $\sum$, and in which first a negative electron in the state $(-2^{1/2}\nu_0, P, \tau)$ jumps to a state near the state (0), and then the original positive electron jumps down from the state (0) to the state $(-2^{1/2}\nu_0, P, \tau)$; in either transition either of the two quanta may be emitted.

If we try to correct (13) to take account of the fact that the energy radiated should be $(m+M_P)c^2$ and not $2mc^2$, we get in place of (14)

$$T' = \frac{(m + M_p)^2c^3}{64\pi^5e^4n_p} \sim \frac{5\times 10^{16}}{n_p} \text{ sec.} \tag{15}$$

Both (14) and (15) give an absurdly short mean life time for matter. With $n_P = 10^{25}$ we get

$$T \sim 5 \times 10^{-15}; \quad T' \sim 5 \times 10^{-9}.$$

Of course the protons and electrons of matter are not in general free, nor uniformly distributed, nor at rest. But we should hardly expect their agglomeration into nuclei, or even atoms, to reduce appreciably the mean transition probability, since this would mean essentially an increase in the effective n_P to be used. In any case (14) or (15) should apply roughly to electrons and protons in a discharge tube.

II

In their paper on the relativistic treatment of the interaction of radiation and matter, Heisenberg and Pauli point out that according to their theory the radiationless transitions of the quantum mechanics may always be expected to be accompanied by transitions which correspond to a change in the material system and the emission of at least one quantum of light.[3] So, for example, in the Auger jumps, in the ionization of an atom in an electric field, in the capture of electrons by alpha-particles, and in the radioactive decay of nuclei, the energy of the liberated particles should show a certain diffuseness; and energy is only conserved by the emission of an appropriate continuous distribution of light. Heisenberg and Pauli derive an expression for the probability of such transitions involving radiation; they show that this probability is small, compared with that of the radiationless transitions, of the order

$$e^2/hc\ \ (v/c)^2.$$

They apply this result to the problem of radioactive disintegration, where the escape of the alpha and beta particles may be roughly schematized as a diffusion through a high wall of potential energy; and they obtain so an explanation of the sharp definition of the energies of alpha-particles, and the great diffuseness of the beta spectrum. The non-appearance of the gamma radiation, which, on this theory, should accompany beta-ray disintegration, remains unexplained.

In this note we shall compute the relative probability of such radiative transitions on the basis of the Dirac radiation theory. For this probability we obtain an expression which, in the approximation to which the calculations of H. P. were carried, should agree with the results of that calculation. In our formula certain terms appear which were deliberately neglected in H. P.; and further this formula is applicable to a slightly more general group of problems than that of H. P., which cannot strictly be applied to any of the problems mentioned above except that of the Auger jumps; but except for these minor modifications our result reduces to that of H. P.; and it gives the same predictions when applied to the theory of radioactive disintegration. The present work is rather simpler than that on the basis of the more general theory.

For the occurrence of radiationless transitions it is essential that the material system (and we shall call this the "atom") be in a quasistationary state of an energy equal to the energy of some aperiodic motion of the system. Let the wave equation for the atom,—which may be written in the configuration space, and without explicit reference to the radiation field, to the order $(v/c)^2$—be

$$(H - h\nu)\psi_\nu = 0. \tag{16}$$

Let the initial state have the energy

$$E_0 = h\nu_0 \tag{17}$$

and be given by a wave packet which satisfies the equation

$$(H - V - h\nu_0)\psi_0 = 0. \tag{18}$$

[3] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929); cited as H. P.

The wave packets for the quasistationary aperiodic states of the atom we call θ_ν; they satisfy

$$(H - V' - h\nu)\theta_\nu = 0;\ E = h\nu \tag{19}$$

and shall be normalized to $d\nu$. Then the probability of a radiationless transition to the continuum is given, to the first order in the small quantity λ_0/ν_0, by

$$\lambda_0 = 4\pi^2/h^2 \mid V_{\nu 0} \mid^2;\ V_{\nu 0} = \int d\tau \delta_{\nu_0} V \psi_0. \tag{20}$$

(The integral over $d\tau$ is to be taken over the configuration space of the atom.)

Now let there be no radiation present initially. Since the atom has energy levels lower than E_0, it can make radiative transitions to these states; Dirac's radiation theory gives us, for the probability per unit time per unit frequency ν_s of the radiation, for this transition

$$\lambda_s d\nu_s = 16\pi^2 \nu_s d\nu_s / 3hc^3 \mid \dot{P}_{\nu_0 - \nu_s, 0} \mid^2$$

with

$$\dot{P}_{\nu_0 - \nu_s, 0} = \int d\tau \bar{\theta}_{\nu_0 - \nu_s} \dot{P} \psi_0 \tag{21}$$

where $\dot{\mathrm{P}}$ is the time rate of change of the electric moment of the atom. (With Dirac's linear Hamiltonian for the electron it will be

$$\sum_R e_k \mathbf{a}^k$$

where e_k is the charge on the k'th particle, and the $\alpha^k s$ are the Dirac matrices, and the summation is to be taken over all particles.)

Now when V is not very small, (21) gives us only a very poor approximation for the probability of the corresponding transitions. Somewhat inaccurately we may say that the system can reach the same final state by a double jump, in which, e.g., the atom goes over into some arbitrary state in the continuum, and then—but there is no interval between the jumps,—jumps to the final state and emits a quantum. In this process of course only the total energy of the system is conserved, and that only when one considers the double jump as a single process. This is the effect treated by H. P.; and to obtain it we have only to carry the perturbation theory a step farther than was necessary for the derivation of (20) and (21).

The Dirac wave equation for the probability amplitude ϕ for the whole system, taken as a function of the state, which for brevity we describe by the single index ν, of the atom, and the number N_{ps} of quanta of frequency ν_s, polarization p, and given direction of propagation, is

$$
\begin{aligned}
-h/2\pi i\cdot\partial/\partial t\phi(\nu,N_{sp}) &= \sum_{\nu' N'_{sp}} H(\nu,N_{sp};\nu' N'_{sp})\phi(\nu',N'_{sp})\\
&= \int dv' V'_{\nu\nu'}\phi(\nu',N_{sp}) + V_{\nu 0}\phi(0,N_{sp})\\
&+ \int d\nu' \sum_{s'p'} N^{1/2}_{s'p'}\nu^{1/2}_{s'}\kappa^{1/2}_{s'p'}(\mathbf{e}_{s'p'}\cdot\dot{\boldsymbol{P}}_{\nu\nu'})\phi(\nu',N_{sp}-\delta_{ss'}\delta_{pp'})\\
&+ \int d\nu' \sum_{s'p'} (N_{s'p'}+1)^{1/2}\nu^{1/2}_{s'}\kappa^{1/2}_{s'p'}(\mathbf{e}_{s'p'},\dot{\boldsymbol{P}}_{\nu\nu'})\phi(\nu',N_{sp}+\delta_{ss'}\delta_{pp'})
\end{aligned}
\tag{22}
$$

with
$$\kappa_{sp}=h/2\pi c^3\sigma_{sp}.$$

Here σ_{sp} is the number of components of the radiation field of given polarization per unit frequency about ν_s per unit solid angle for the direction of propagation; and $\mathbf{e}_p$ is a unit vector parallel to the electric vector of the component sp. The summation $\sum_{\mathrm{sp}}$, and the product Π_{sp} infra, are to be taken over all the components of the field.

If we take for our initial conditions for ϕ

$$
\begin{aligned}
\phi(\nu,N_{sp}) &= 0\\
\phi(0,N_{sp}) &= \prod_{sp}\delta(N_{sp},0)e^{-2\pi i\nu_0 t}
\end{aligned}
\tag{23}
$$

and put these values in (7), we find in first approximation

$$
\begin{aligned}
\phi_1(\nu,N_{sp}) &= \prod_{sp}\delta(N_{sp},0)V_{\nu_0}\frac{e^{-2\pi i\nu t}-e^{-2\pi i\nu_0 t}}{h(\nu-\nu_0)}\\
&+ \sum_{s'p'}\delta(N_{s'p'},1)\prod_{s''p''}{}'\delta(N_{s''p''},0)\nu^{1/2}_{s'}x^{1/2}_{s'p'}(\mathbf{e}_{s'p'}\cdot\dot{\boldsymbol{P}}_{\nu_0})\frac{e^{-2\pi i(\nu+\nu_{s'})t}-e^{-2\pi i\nu_0 t}}{h(\nu+\nu_{s'}-\nu_0)}
\end{aligned}
\tag{24}
$$

In Π' the factor for $s''=s'$, $p''=p'$ is to be omitted.
If one puts this expression (24) for ϕ_1 in (22), integrates the equation to obtain the second approximation $\phi_2(\nu, N_{sp})$, and takes the sum over all components of the field

$$\sum_p \int d\nu_s \int d\omega_s \,|\,\phi_2(\nu,1_{sp})\,|^2,$$

this gives the probability that the system has, at time t emitted a quantum of frequency near $\nu_s=\nu_0-\bar{\nu}$, and made a transition to a state in the continuum of energy near $h\bar{\nu}$. The coefficient of t in this expression gives the transition probability for transitions to a state in the range $\bar{\nu}$ to $\bar{\nu}+\Delta\bar{\nu}$:

$$\lambda_s(\bar{\nu})\Delta\bar{\nu} = \frac{16\pi^2\nu_s\Delta\bar{\nu}}{3c^3h^3}\left| h\dot{\boldsymbol{P}}_{0\nu} - \int\frac{d\nu' V_{0\nu'}\dot{\boldsymbol{P}}_{\nu'\bar{\nu}}}{\nu_0-\nu'} - \int\frac{d\nu'\,\dot{\boldsymbol{P}}_{0\nu'}V'_{\nu'\bar{\nu}}}{\bar{\nu}-\nu'}\right|^2. \tag{25}$$

The first term in the bracket is the direct emission given by (21) and neglected by H. P.; the remaining terms differ from those of H. P. (132) only by having

V' in place of V. When ψ_0 and θ_ν are characteristic functions of the same equation, these terms reduce to those given in H. P.

It should be observed that both (25) and H. P. (132) are derived as approximations; in particular, the momentum of the light quantum, and terms of higher order in v/c, are neglected in both computations. The retention of this momentum leads to the same modification in (25) and H. P. (132); and so does the retention or terms of the fourth order in v/c. But in higher orders only the method of H. P. can be used, since then no equation of the form (16) holds for the atom, and it is necessary to work directly from the more general equations of quantum electrodynamics, and take more complete account of the retardation of the forces between the particles of the atom.

To obtain the order of magnitude of the total radiative transition probability for radioactive disintegrations, we may observe that V and V' may be expected to be of the same order of magnitude as $h\nu_0$, and we integrate (25) for all frequencies ν_s up to ν_0; this gives

$$\lambda_s = \int_0^{\nu_0} \lambda_s(\bar{\nu})d\bar{\nu} \sim v^2\nu_0{}^2e^2/c^3h \left| \int d\tau\bar{\theta}_\nu\psi_0 \right|^2. \tag{26}$$

The ratio of this to (20) gives the relative probability that radiation will be emitted in the disintegration:

$$\lambda_s/\lambda_0 \sim e^2/hc\cdot(v/c)^2 \tag{27}$$

in agreement with H. P. (133). From (25) it is clear that only a much more detailed knowledge of θ_ν and ψ_0 and of the form of the Hamiltonian H in (16) than is at present available can give us any precise value for λ_s/λ_0.

The application of (27) enables us to estimate the relative probability of radiative and radiationless capture of electrons from atoms by an alpha particle, and gives for the ratio of the probabilities 10^{-6}, in agreement with the more detailed calculations of the effect.